Orthogonality - Subordination - Freeness Convolutions and Graph Products

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Topics

1. Orthogonal structures.
2. Subordinate structures.
3. Decompositions of the free additive convolution.
4. Orthogonal product of graphs.
5. Subordinate product of graphs and its decomposition.
6. Decompositions of the free product of graphs.

Motivations

1. Find relations between noncommutative probability theories associated with the main notions of noncommutative independence:
    1. tensor
    2. free
    3. boolean
    4. monotone
2. Study relations between the associated convolutions of probability measures.
3. Associate products of rooted graphs with the main notions of noncommutative independence: Cartesian, free, star, comb
4. Find relations between these products.

Noncommutative independence

Assummptions:

1. $\mathcal{A}_1$, $\mathcal{A}_2$ be subalgebras of a unital algebra $\mathcal{A}$
2. φ – normalized linear functional on $\mathcal{A}$.
3. $a_1, a_2, \ldots \in \mathcal{A}_1$
4. $b_1, b_2, \ldots \in \mathcal{A}_2$

Types of independence

1. *tensor independence*

$$\varphi(a_1 b_1 a_2 b_2 \ldots) = \varphi(a_1 a_2 \ldots)\varphi(b_1 b_2 \ldots)$$

2. *free independence* (Voiculescu 1985, Avitzour 1982)

$$\varphi(a_1 b_1 a_2 b_2 \ldots) = 0$$

   whenever $a_i, b_i \in \ker\varphi$ for every i

3. *boolean independence* (regular free – Bożejko 1986)

$$\varphi(a_1 b_1 a_2 b_2 \ldots) = \varphi(a_1)\varphi(b_1)\varphi(a_2)\varphi(b_2)\ldots$$

4. *monotone independence* (Muraki 2001)

$$\varphi(a_1 b_1 a_2 b_2 \ldots) = \varphi(a_1 a_2 \ldots)\varphi(b_1)\varphi(b_2)\ldots$$

Rooted graphs

Basic definitions:

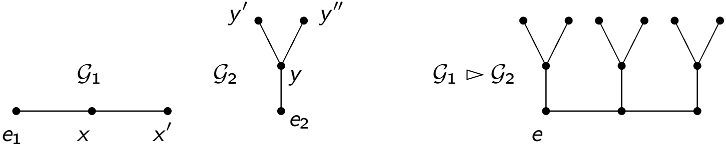
1. By a *rooted graph* we understand a pair $(\mathcal{G}, e)$, where $\mathcal{G} = (V, E)$ is a locally finite non-oriented connected simple graph (no loops, no multiple edges) and e is a selected vertex.
2. By the *spectral distribution* of $(\mathcal{G}, e)$ we understand the distribution of the adjacency matrix $A(\mathcal{G})$ in the state φ associated with the root e.

Comb product

Definition

The *comb product* of rooted graphs $(\mathcal{G}_1, e_1)$ and $(\mathcal{G}_2, e_2)$ is the rooted graph $(\mathcal{G}_1 \rhd \mathcal{G}_2, e)$ obtained by attaching a copy of $\mathcal{G}_2$ by its root e_2 to each vertex of $\mathcal{G}_1$, where e is the vertex obtained by identifying e_1 and e_2.

Example



Monotone decomposition of the comb product

Decomposition Theorem [Accardi, Ben Ghorbal, Obata]

Let $(\mathcal{G}_1, e_1)$ and $(\mathcal{G}_2, e_2)$ be rooted graphs with spectral distributions μ and ν, respectively. Then, the adjacency matrix of their comb product can be decomposed as

$$A(\mathcal{G}_1 \rhd \mathcal{G}_2) = A^{(1)} + A^{(2)}$$

where $A^{(1)}$ and $A^{(2)}$ are monotone independent w.r.t. $\varphi(.) = \langle .\delta(e), \delta(e)\rangle$. Moreover, the spectral distribution of $(\mathcal{G}_1 \rhd \mathcal{G}_2, e)$ is given by $\mu \rhd \nu$.

Tensor product realization

Tensor product realization

The summands in the comb product can be identified with

$$A^{(1)} = A_1 \otimes 1, \quad A^{(2)} = P_1 \otimes A_2$$

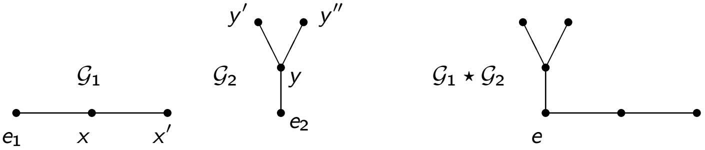
where A_1 and A_2 are the adjacency matrices of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and P_1 is the projection onto $\mathbb{C}\delta(e_1)$.

Star product

Definition

The *star product* of $(\mathcal{G}_1, e_1)$ and $(\mathcal{G}_2, e_2)$ is the graph $(\mathcal{G}_1 \star \mathcal{G}_2, e)$ obtained by attaching a copy of $\mathcal{G}_2$ by its root e_2 to the root e_1 of $\mathcal{G}_1$, where e is the vertex obtained by identifying e_1 and e_2.

Example


$\mathcal{G}_1$
e_1
x
x'
y'
y''
$\mathcal{G}_2$
y
e_2
$\mathcal{G}_1 \star \mathcal{G}_2$
e

Boolean decomposition of the star product

Decomposition Theorem [R.L., Obata]

Let $(\mathcal{G}_1, e_1)$ and $(\mathcal{G}_2, e_2)$ be rooted graphs with spectral distributions μ and ν, respectively. Then, the adjacency matrix of their star product can be decomposed as

$$A(\mathcal{G}_1 \star \mathcal{G}_2) = A^{(1)} + A^{(2)}$$

where $A^{(1)}$ and $A^{(2)}$ are boolean independent w.r.t. φ, where $\varphi(.) = \langle .\delta(e), \delta(e)\rangle$. Moreover, the spectral distribution of $(\mathcal{G}_1 \star \mathcal{G}_2, e)$ is given by the boolean convolution $\mu \uplus \nu$.

Tensor product realization

Tensor realization of star product

The summands in the star product can be identified with

$$A^{(1)} = A_1 \otimes P_2, \quad A^{(2)} = P_1 \otimes A_2$$

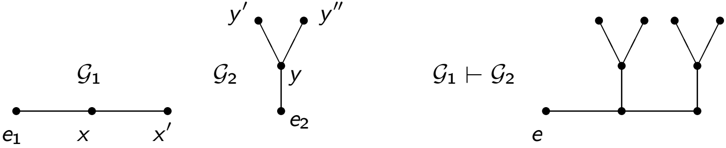
where P_i is the projection onto $\mathbb{C}\delta(e_i)$.

Orthogonal product

Definition

The *orthogonal product* of two rooted graphs $(\mathcal{G}_1, e_1)$ and $(\mathcal{G}_2, e_2)$ is the rooted graph $(\mathcal{G}_1 \vdash \mathcal{G}_2, e)$ obtained by attaching a copy of $\mathcal{G}_2$ by its root e_2 to each vertex of $\mathcal{G}_1$ but the root e_1, where e is taken to be equal to e_1.

Example


$\mathcal{G}_1$
e_1
x
x'
$\mathcal{G}_2$
y'
y''
y
e_2
$\mathcal{G}_1 \vdash \mathcal{G}_2$
e

Definition

Let $(\mathcal{A}, \varphi, \psi)$ be a unital algebra with a pair of linear normalized functionals and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be non-unital subalgebras of $\mathcal{A}$. We say that $\mathcal{A}_2$ is *orthogonal* to $\mathcal{A}_1$ with respect to (φ, ψ) if

1. $\varphi(bw_2) = \varphi(w_1 b) = 0$
2. $\varphi(w_1 a_1 b a_2 w_2) = \psi(b)\left(\varphi(w_1 a_1 a_2 w_2) - \varphi(w_1 a_1)\varphi(a_2 w_2)\right)$ for any $a_1, a_2 \in \mathcal{A}_1$, $b \in \mathcal{A}_2$ and any $w_1, w_2 \in \mathrm{alg}(\mathcal{A}_1, \mathcal{A}_2)$

We say that the pair (a, b) of elements of $\mathcal{A}$ is *orthogonal* with respect to (φ, ψ) if the algebra generated by $a \in \mathcal{A}$ is orthogonal to the algebra generated by $b \in \mathcal{A}$.

Orthogonal convolution

Definition

Let $(a, b) \in \mathcal{A}$ be an orthogonal pair of elements of $\mathcal{A}$, μ – the φ-distribution of a and ν – the ψ-distribution of b. By the *orthogonal convolution* of μ and ν, denoted $\mu \vdash \nu$, we understand the φ-distribution of $a + b$.

Moments

Moments of low orders:

$$
\begin{aligned}
(\mu \vdash \nu)(1) &= \mu(1) \\
(\mu \vdash \nu)(2) &= \mu(2) \\
(\mu \vdash \nu)(3) &= \mu(3) + (\mu(2) - \mu^2(1))\nu(1) \\
(\mu \vdash \nu)(4) &= \mu(4) + 2\mu(3)\nu(1) + \mu(2)\nu(2) \\
&\quad -2\mu(2)\mu(1)\nu(1) - \mu^2(1)\nu(2)
\end{aligned}
$$

Decomposition of the orthogonal product

Decompositon Theorem [R.L.]

Let $(\mathcal{G}_1, e_1)$ and $(\mathcal{G}_2, e_2)$ be rooted graphs with spectral distributions μ and ν, respectively. Then

1. the adjacency matrix of their orthogonal product can be decomposed as

$$A(\mathcal{G}_1 \vdash \mathcal{G}_2) = A^{(1)} + A^{(2)}$$

   where the pair $(A^{(1)}, A^{(2)})$ is orthogonal w.r.t. (φ, ψ), with φ and ψ states associated with vectors $\delta(e), \delta(v) \in l_2(V_1 \vdash V_2)$ and $v \in V_1^0$, where $V_1^0 = V_1 \backslash \{e_1\}$,

2. the spectral distribution of $(\mathcal{G}_1 \vdash \mathcal{G}_2, e)$ is given by $\mu \vdash \nu$.

Tensor product realization

Tensor realization of orthogonal product

The summands in the orthogonal product can be identified with

$$A^{(1)} = A_1 \otimes P_2, \quad A^{(2)} = P_1^{\perp} \otimes A_2$$

where P_i is the projection onto $\mathbb{C}\delta(e_i)$.

Transforms

Transforms of probability measures on the real line

1. the *Cauchy-transform* of μ:

$$G_\mu(z) = \sum_{n=0}^{\infty} m_\mu(n) z^{-n-1} = \int_{\mathbf{R}} \frac{d\mu(t)}{z-t}.$$

2. the *R-transform* of μ:

$$R_\mu(z) = G_\mu^{-1}(z) - \frac{1}{z}$$

Transforms

Transforms of probability measures on the real line

1. the *K-transform* of μ:

$$K_\mu(z) = z - \frac{1}{G_\mu(z)}$$

2. the *reciprocal Cauchy transform* of μ:

$$F_\mu(z) = \frac{1}{G_\mu(z)}$$

Transforms vs convolutions

Transforms associated with convolutions

The following transforms are important:

1. $R_\mu(z)$ – *free additive convolution* $\mu \boxplus \nu$ – addition formula (Voiculescu)
2. $K_\mu(z)$ – *boolean convolution* $\mu \uplus \nu$– addition formula (Speicher Woroudi)
3. $F_\mu(z)$ – *monotone convolution* $\mu \rhd \nu$ – composition formula (Muraki)

Orthogonal convolution in terms of transforms

Theorem [R.L.]

Let μ and ν be probability measures on the real line. The reciprocal Cauchy transform of $\mu \vdash \nu$ is given by the formula

$$F_{\mu \vdash \nu}(z) = F_{\mu}(F_{\nu}(z)) - F_{\nu}(z) + z$$

Equivalently, we have

$$K_{\mu \vdash \nu}(z) = K_{\mu}(F_{\nu}(z)) = K_{\mu}(z - K_{\nu}(z))$$

Remark

Compare with the monotone convolution

$$F_{\mu \rhd \nu}(z) = F_{\mu}(F_{\nu}(z))$$

established by Muraki.

Free product of rooted graphs

Definition

By the *free product* of rooted sets $(V_1, e_1) * (V_2, e_2)$ we understand the rooted set $(V_1 * V_2, e)$, where

$$V_1 * V_2 = \{e\} \cup \{v_1 v_2 \dots v_m;\ v_k \in V_{i_k}^0\ \wedge\ i_1 \neq i_2 \neq \dots \neq i_n,\ m \in \mathbb{N}\}$$

and e is the empty word.

Definition

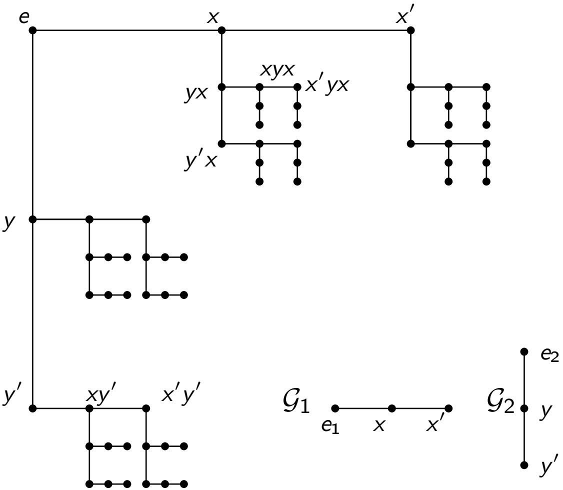
By the *free product* of rooted graphs $(\mathcal{G}_1, e_1) * (\mathcal{G}_2, e_2)$, or simply $\mathcal{G}_1 * \mathcal{G}_2$, we understand the rooted graph $(V_1 * V_2, E_1 * E_2, e)$ where

$$E_1 * E_2 = \{\{vu, v'u\} :\ \{v, v'\} \in \bigcup_{i=1,2} E_i \text{ and } u, vu, v'u \in V_1 * V_2\}.$$

Definition

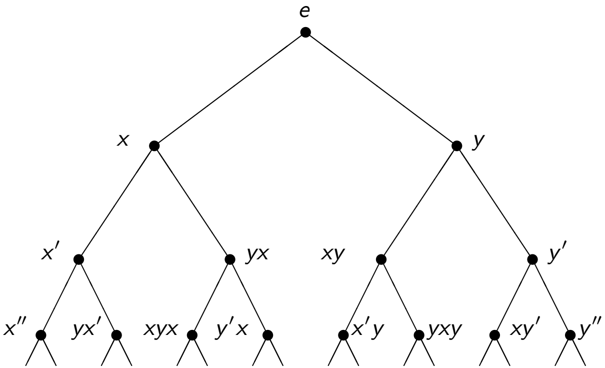
By the *m-free product* of rooted graphs we understand the subgraph of $\mathcal{G}_1 * \mathcal{G}_2$ obtained by restricting the set of vertices to words of lenght $\leqslant m$ with the root e.
Notation: $(\mathcal{G}_1, e_1) *^{(m)} (\mathcal{G}_2, e_2)$ or $(\mathcal{G}_1 *^{(m)} \mathcal{G}_2, e)$.

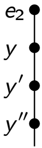
Example

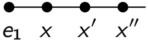


4-free product $(\mathcal{G}_1, e_1) *^{(4)} (\mathcal{G}_2, e_2)$ with selected vertices labelled.

Binary tree

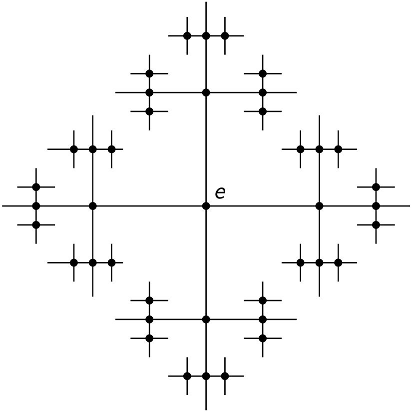


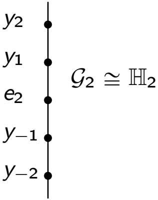


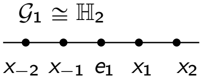


$$\mathbb{T}_2 \cong \mathbb{T}_1 * \mathbb{T}_1$$

Homogenous tree







$$\mathbb{H}_4 \cong \mathbb{H}_2 * \mathbb{H}_2.$$

Free decomposition of the free product

Notation:
$W_i(n) = \{v_1 v_2 \ldots v_n \in V_1 * V_2 : \;\; v_1 \notin V_i^0\}$
$P_i(n) : \; l_2(V_1 * V_2) \to l_2(W_i(n)), \; n \geqslant 1$
$P_i(0) : \; l_2(V_1 * V_2) \to \mathbb{C}\delta(e), \; i = 1, 2.$
φ – vacuum expectation on $l_2(V_1 * V_2)$

Decomposition Theorem

Decomposition Theorem [Accardi, R.L., Salapata]

The adjacency matrix $A(\mathcal{G}_1 * \mathcal{G}_2)$ admits a decomposition of the form

$$A(\mathcal{G}_1 * \mathcal{G}_2) = A^{(1)} + A^{(2)}$$

where $A^{(1)}$ and $A^{(2)}$ are free w.r.t. φ. Moreover,

$$A^{(i)} = \sum_{n=1}^{\infty} A_i(n) = \sum_{n=1}^{\infty} A_i P_i(n-1)$$

where the action of A_i is given by $A_i \delta(xu) = \delta(x'u)$ whenever $\{x, x'\} \in E_i$ for $i = 1, 2$.

Spectral distribution

Corollary

Let μ and ν denote spectral distributions of $(\mathcal{G}_1, e_1)$ and $(\mathcal{G}_2, e_2)$, respectively. Then the spectral distribution of $(\mathcal{G}_1 * \mathcal{G}_2, e)$ is given by $\mu \boxplus \nu$ (well-known fact, but here it follows from the decomposition).

Subordinate branches

Definition

By the *branch of* $(V_1, e_1) * (V_2, e_2)$ *subordinate to* (V_j, e_j), we shall understand the rooted set (S_j, e), where

$$S_j = \{e\} \cup \{v_1 v_2 \ldots v_m \in V_1 * V_2 : \ \ v_m \in V_j^0, \ m \in \mathbb{N}\}$$

By the *branch of* $(\mathcal{G}_1 * \mathcal{G}_2, e)$ *subordinate to* $(\mathcal{G}_j, e_j)$, where $j = 1, 2$, we shall understand the rooted graph $(\mathcal{B}_j, e)$, where $\mathcal{B}_j$ is the subgraph of $\mathcal{G}_1 * \mathcal{G}_2$ restricted to the set S_j. The concept of 'branches' is due to Quenell.

Binary tree as a subordinate branch

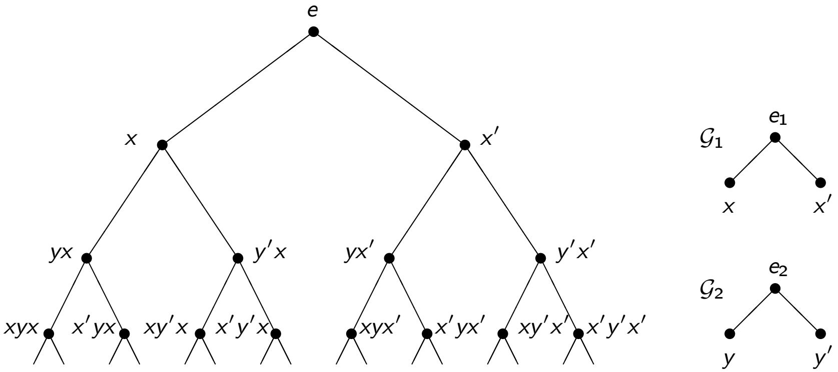


Fig. 5. $\mathbb{T}_2 \cong \mathcal{B}_1$.

Assumptions:

1. $(\mathcal{A}, \varphi, \psi)$ is a unital algebra with a pair of linear normalized functionals,
2. $\mathcal{A}_1$ is a unital subalgebra of $\mathcal{A}$,
3. $\mathcal{A}_2$ is a non-unital subalgebra with an 'internal' unit 1_2, i.e. $1_2 b = b = b 1_2$ for every $b \in \mathcal{A}_2$,
4. $\mathcal{A}_1^0 = \mathcal{A}_1 \cap \ker\varphi$,
5. $\mathcal{A}_2^0 = \mathcal{A}_2 \cap \ker\psi$.

Freeness with subordination

Definition

We say that the pair $(\mathcal{A}_1, \mathcal{A}_2)$ is *free with subordination*, or simply *s-free*, with respect to (φ, ψ) if $\psi(1_2) = 1$ and it holds that

1. $\varphi(a_1 a_2 \ldots a_n) = 0$ whenever $a_j \in \mathcal{A}_{i_j}^0$ and $i_1 \neq i_2 \neq \ldots \neq i_n$
2. $\varphi(w_1 1_2 w_2) = \varphi(w_1 w_2) - \varphi(w_2)\varphi(w_2)$ for any $w_1, w_2 \in \mathrm{alg}(\mathcal{A}_1, \mathcal{A}_2)$.

We say that the pair (a, b) of random variables from $\mathcal{A}$ is *s-free* with respect to (φ, ψ) if the pair of algebras generated by these random variables is s-free with respect to (φ, ψ).

Remark

This notion leads to the algebraic (and operator) approach to the concept of analytic subordination in free probability studied by Voiculescu, Biane, Bercovici, Belinschi, Chistyakov, Goetze.

Decomposition of adjacency matrices

Decomposition Theorem

The adjacency matrix of the branch $\mathcal{B}_1$ can be decomposed as $A(\mathcal{B}_1) = A^{(1)} + A^{(2)}$, where the strongly convergent series

$$A^{(1)} = \sum_{n \text{ odd}} A_1(n), \quad A^{(2)} = \sum_{n \text{ even}} A_2(n),$$

are s-free w.r.t. (φ, ψ), where $\varphi(.) = \langle .\delta(e), \delta(e)\rangle$ and $\psi(.) = \langle .\delta(v), \delta(v)\rangle$ and $v \in V_1^0$. An analogous decomposition holds for the branch $\mathcal{B}_2$ with the summations over odd and even n interchanged.

Orthogonal decomposition of subordinate branches

Theorem [R.L.]

The branch $\mathcal{B}_1$ can be decomposed as

$$\mathcal{B}_1 = \mathcal{G}_1 \vdash (\mathcal{G}_2 \vdash (\mathcal{G}_1 \vdash \ldots)))$$

An analogous statement holds for the branch $\mathcal{B}_2$.

Corollary

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are uniformly locally finite, the spectral distribution of $\mathcal{B}_1$ is given by the so-called s-free convolution

$$\mu \boxplus \nu := \mu \vdash (\nu \vdash (\mu \vdash \ldots)))$$

(the right-hand side is understood as the weak limit). An analogous statement holds for the branch $\mathcal{B}_2$.

Corollary

The K-transform of $\mu \boxvdash \nu$ can be expressed in the 'continued composition form'

$$K_{\mu \boxvdash \nu}(z) = K_{\mu}(z - K_{\nu}(z - K_{\mu}(z - K_{\nu}(\ldots))))$$

where the right-hand side is understood as the uniform limit on compact subsets of the complex upper half-plane.

Example

Consider two rooted graphs $\mathcal{G}_1$, $\mathcal{G}_2$, whose spectral distributions μ, ν are associated with reciprocal Cauchy transforms

$$F_\mu(z) = z - \alpha_0 - \frac{\omega_0}{z - \alpha_1}, \quad F_\nu(z) = z - \beta_0 - \frac{\gamma_0}{z - \beta_1}.$$

Then

$$K_{\mu \boxplus \nu}(z) = \alpha_0 + \cfrac{\omega_0}{z - \alpha_1 - \beta_0 - \cfrac{\gamma_0}{z - \alpha_0 - \beta_1 - \cfrac{\omega_0}{z - \alpha_1 - \beta_0 - \cfrac{\gamma_0}{\ldots}}}}$$

and thus, the distribution $\mu \boxplus \nu$ of branch $\mathcal{B}_1$ is associated with the sequences of Jacobi parameters

$$\alpha = (\alpha_0, \alpha_1 + \beta_0, \alpha_0 + \beta_1, \alpha_1 + \beta_0, \ldots), \quad \omega = (\omega_0, \gamma_0, \omega_0, \gamma_0, \ldots)$$

which correspond to the so-called *mixed periodic Jacobi continued fraction* of Kato.

Partial decompositions of free products

Theorem [Quenell]

The free product of rooted graphs admits the star decomposition

$$\mathcal{G}_1 * \mathcal{G}_2 \cong \mathcal{B}_1 \star \mathcal{B}_2$$

and the comb decompositions

$$\mathcal{G}_1 * \mathcal{G}_2 \cong \mathcal{G}_1 \rhd \mathcal{B}_2 \cong \mathcal{G}_2 \rhd \mathcal{B}_1$$

Corollary

The following relations hold:

$$
\begin{aligned}
F_{\mu \boxplus \nu}(z) &= F_{\mu}(F_{\nu \boxvdash \mu}(z)) + F_{\nu}(F_{\mu \boxvdash \nu}(z)) - z \\
F_{\mu \boxplus \nu}(z) &= F_{\mu}(F_{\nu \boxvdash \mu}(z)) = F_{\nu}(F_{\mu \boxvdash \nu}(z))
\end{aligned}
$$

where μ and ν are spectral distributions of rooted graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.

Theorem

The free product of rooted graphs admits the decomposition

$$\mathcal{G}_1 * \mathcal{G}_2 \cong (\mathcal{G}_1 \vdash (\mathcal{G}_2 \vdash (\mathcal{G}_1 \vdash \ldots))) \star (\mathcal{G}_2 \vdash (\mathcal{G}_1 \vdash (\mathcal{G}_2 \vdash \ldots))).$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are uniformly locally finite, its spectral distribution is given by the weak limit

$$\mu \boxplus \nu = (\mu \vdash (\nu \vdash (\mu \ldots))) \uplus (\nu \vdash (\mu \vdash (\nu \ldots)))$$

and thus

$$\mu \boxplus \nu = (\mu \boxvdash \nu) \uplus (\nu \boxvdash \mu)$$

K-transform decomposition

Remark

The K-transform of $\mu \boxplus \nu$ can be expressed in the 'continued composition form'

$$\begin{aligned} K_{\mu\boxplus\nu}(z) &= K_\mu(z - K_\nu(z - K_\mu(z - K_\nu(\ldots)))) \\ &\quad + K_\nu(z - K_\mu(z - K_\nu(z - K_\mu(\ldots)))) \end{aligned}$$

where the right-hand side is understood as the uniform limit on compact subsets of the complex upper half-plane.

Complete decomposition of free products of type II

Theorem

The free product of rooted graphs admits the decomposition

$$\mathcal{G}_1 * \mathcal{G}_2 \cong \mathcal{G}_1 \rhd (\mathcal{G}_2 \vdash (\mathcal{G}_1 \vdash (\mathcal{G}_2 \vdash \ldots)))$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are uniformly locally finite, its spectral distribution is given by the weak limit

$$\mu \boxplus \nu = \mu \rhd (\nu \vdash (\mu \vdash (\nu \vdash (\mu \ldots))))$$

and thus

$$\mu \boxplus \nu = \mu \rhd (\nu \boxplus \mu).$$

Cauchy transform decomposition

Remark

The Cauchy transform of $\mu \boxplus \nu$ can be expressed in the 'continued composition form'

$$G_{\mu \boxplus \nu}(z) = G_\mu(z - K_\nu(z - K_\mu(z - K_\nu(\ldots))))$$

where the right-hand side is understood as the uniform limit on compact subsets of the complex upper half-plane.

Concluding remarks

1. We have completed the scheme in which products of rooted graphs are associated with the main notions of noncommutative independence.
2. We obtained 'complete' decompositions of two types of the free product of graphs and of the free additive convolution.
3. This method (especially the decomposition of type II) allows to compute $\mu \boxplus \nu$ without using the R-transforms in certain cases.
4. The approximants of these decompositions give the so-called *hierarchy of freeness* [R.L.] and the *monotone hierarchy of freeness* [R.L. & Salapata].
5. Our approach contributes new elements to the operator approach to the concept of subordination.

Papers

1. R. Lenczewski, *Decompositions of the free additive convolution*, arXiv: math.OA/0608236, 2006.
2. L. Accardi, R. Lenczewski, R. Salapata, *Decompositions of the free product of graphs*, arXiv: math.CO/0609329, 2006.